

Inversion Formula for the Windowed Fourier Transform*

Wenchang Sun

Department of Mathematics and LPMC, Nankai University, Tianjin 300071, China

Email: sunwch@nankai.edu.cn

Abstract

In this paper, we study the inversion formula for recovering a function from its windowed Fourier transform. We give a rigorous proof for an inversion formula which is known in engineering. We show that the integral involved in the formula is convergent almost everywhere on \mathbb{R} as well as in L^p for all $1 < p < \infty$ if the function to be reconstructed is.

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1 Introduction and the Main Result

The Fourier transform is a very useful mathematical tool, which has been widely used in characterization of function spaces as well as in signal and image processing [6, 11]. For a function $f \in L^1(\mathbb{R})$, the Fourier transform of f is defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-ix\omega} dx.$$

To study local properties of functions (signals), the windowed Fourier transform, also known as short-time Fourier transform, is introduced.

Given a window function $g(x)$, the windowed Fourier transform of a function f with respect to g is defined by

$$(F_g f)(t, \omega) = \int_{\mathbb{R}} f(x) \overline{g(x-t)} e^{-ix\omega} dx.$$

It is easy to see that $F_g f$ is well defined if $f \in L^p(\mathbb{R})$ and $g \in L^{p'}(\mathbb{R})$, where $p, p' \geq 1$ and $1/p + 1/p' = 1$.

Continuous and discrete windowed Fourier transforms have been discussed extensively in the literature since they are widely used in communication theory, quantum mechanics, and many other fields. We refer to [3, 4, 5, 7, 8] for an introduction to the windowed Fourier transform.

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Finding a computationally efficient algorithm for the inversion of windowed Fourier transforms is a fundamental topic in both theory and applications. The classical method to recover f from its windowed Fourier transform is to use the following inversion formula,

$$f(x) = \frac{1}{2\pi\|g\|_2^2} \iint_{\mathbb{R}^2} (F_g f)(t, \omega) g(x-t) e^{ix\omega} dt d\omega, \quad (1.1)$$

where we assume that $g \in L^2(\mathbb{R})$. It can be shown that the convergence is in $L^2(\mathbb{R})$ as well as in many other spaces if the function to be reconstructed is and g satisfies some further conditions [7].

Since a double integral is involved in (1.1), it is obviously very complicated. An alternate method is to use the filter-bank summation [1],

$$f(x) = \frac{1}{2\pi g(0)} \int_{\mathbb{R}} (F_g f)(x, \omega) e^{ix\omega} d\omega, \quad (1.2)$$

where we assume that $g(0) \neq 0$. Note that (1.2) was presented in [1] in a discrete version for compactly supported window functions and the authors stated that their results may be equally well stated in a continuous time-domain setting.

Although (1.2) is well known in engineering, the convergence of the integral is not well stated in literature. In this paper, we show that the integral in (1.2) is convergent in $L^p(\mathbb{R})$ for all $1 < p < \infty$ if the function f is. Moreover, by applying the Carleson-Hunt theorem, we also show that the convergence is almost everywhere on \mathbb{R} .

Before stating our result, we introduce some definitions. Throughout this paper, x_0 is a fixed real number. For any $A_1, A_2 > 0$, define

$$(T_{A_1, A_2} f)(x) = \int_{-A_1}^{A_2} (F_g f)(x - x_0, \omega) e^{ix\omega} d\omega. \quad (1.3)$$

Our main result is the following.

Theorem 1.1 *Suppose that g is continuous and that $g, \hat{g} \in L^1(\mathbb{R})$. Then for any $f \in L^p(\mathbb{R})$, $1 < p < \infty$, we have*

$$\lim_{A_1, A_2 \rightarrow \infty} \|T_{A_1, A_2} f - 2\pi \overline{g(x_0)} f\|_p = 0 \quad (1.4)$$

and

$$\lim_{A \rightarrow \infty} (T_A f)(x) = 2\pi \overline{g(x_0)} f(x), \quad a.e., \quad (1.5)$$

where we use the shortcut $T_A f = T_{A, A} f$.

Remark 1.2 *The reconstruction formula (1.3) is stable in the sense that for any $f, \tilde{f} \in L^p(\mathbb{R})$,*

$$\|T_{A_1, A_2}(f - \tilde{f})\|_p \leq 2C_p \|\hat{g}\|_1 \|f - \tilde{f}\|_p, \quad \forall A_1, A_2 > 0,$$

where C_p is a constant depending only on p . For details, see the proof of Theorem 1.1.

In Section 2, we give the proof of Theorem 1.1, which is based on the famous Carleson-Hunt theorem [2, 9] for Fourier series and the extension to Fourier integrals by Kenig and Tomas [10].

2 Proof of the Main Result

In this section, we give the proof of the main result.

We begin with a simple lemma on the Fourier transform, for which we omit the proof.

Lemma 2.1 *For any $f \in L^2(\mathbb{R})$ with $\hat{f} \in L^1(\mathbb{R})$, we have*

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) e^{ix\omega} d\omega, \quad a.e.$$

We also need the following formula on the windowed Fourier transform.

Proposition 2.2 ([8, Lemma 3.1.1]) *For any $f, g \in L^2(\mathbb{R})$, we have*

$$(F_g f)(x, \omega) = \frac{1}{2\pi} (F_{\hat{g}} \hat{f})(\omega, -x) e^{-ix\omega}.$$

Next, we show that for $f \in L^2(\mathbb{R})$ with $\hat{f} \in L^1(\mathbb{R})$, $T_{A_1, A_2} f$ is convergent in L^∞ norm.

Lemma 2.3 *Suppose that g is continuous and that $g, \hat{g} \in L^1(\mathbb{R})$. Then for any $f \in L^2(\mathbb{R})$ with $\hat{f} \in L^1(\mathbb{R})$, we have*

$$\lim_{A_1, A_2 \rightarrow \infty} \|T_{A_1, A_2} f - 2\pi \overline{g(x_0)} f\|_\infty = 0. \quad (2.1)$$

Proof. For any $f \in L^2(\mathbb{R})$, we see from Proposition 2.2 that

$$\begin{aligned} (T_{A_1, A_2} f)(x) &= \int_{-A_1}^{A_2} (F_g f)(x - x_0, \omega) e^{ix\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-A_1}^{A_2} (F_{\hat{g}} \hat{f})(\omega, x_0 - x) e^{-i(x-x_0)\omega} e^{ix\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-A_1}^{A_2} e^{ix_0\omega} d\omega \int_{\mathbb{R}} \hat{f}(y) \overline{\hat{g}(y-\omega)} e^{-iy(x_0-x)} dy \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y) e^{iyx} dy \int_{-A_1}^{A_2} \overline{\hat{g}(y-\omega)} e^{-ix_0(y-\omega)} d\omega \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y) e^{iyx} dy \int_{y-A_2}^{y+A_1} \overline{\hat{g}(\omega)} e^{-ix_0\omega} d\omega \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\hat{g}(\omega)} e^{-ix_0\omega} d\omega \int_{\omega-A_1}^{\omega+A_2} \hat{f}(y) e^{iyx} dy, \end{aligned} \quad (2.2)$$

where we use Fubini's theorem twice. By Lemma 2.1, for almost every x ,

$$\begin{aligned} &(T_{A_1, A_2} f)(x) - 2\pi \overline{g(x_0)} f(x) \\ &= \int_{\mathbb{R}} \overline{\hat{g}(\omega)} e^{-ix_0\omega} d\omega \left(\frac{1}{2\pi} \int_{\omega-A_1}^{\omega+A_2} \hat{f}(y) e^{iyx} dy - f(x) \right) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\hat{g}(\omega)} e^{-ix_0\omega} d\omega \int_{\substack{y < \omega-A_1 \\ \text{or } y > \omega+A_2}} \hat{f}(y) e^{iyx} dy. \end{aligned}$$

Hence

$$\|T_{A_1, A_2}f - 2\pi g(x_0)f\|_\infty \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{g}(\omega)| d\omega \int_{\substack{y < \omega - A_1 \\ \text{or } y > \omega + A_2}} |\hat{f}(y)| dy.$$

By the dominated convergence theorem, we get

$$\lim_{A_1, A_2 \rightarrow \infty} \|T_{A_1, A_2}f - 2\pi \overline{g(x_0)}f\|_\infty = 0.$$

This completes the proof. \square

In the followings we prove the convergence in $L^p(\mathbb{R})$. First, we show that T_{A_1, A_2} is well defined on $L^p(\mathbb{R})$.

Lemma 2.4 *Suppose that g is continuous and that $g, \hat{g} \in L^1(\mathbb{R})$. For any $A_1, A_2 > 0$, let*

$$\begin{aligned} K_{A_1, A_2}(x, y) &= \overline{g(y - x + x_0)} \cdot \left(\frac{\sin A_1(y - x) + \sin A_2(y - x)}{(y - x)} \right. \\ &\quad \left. - i \cdot \frac{2 \sin \frac{(A_2 - A_1)(y - x)}{2} \sin \frac{(A_2 + A_1)(y - x)}{2}}{(y - x)} \right). \end{aligned} \quad (2.3)$$

Then we have

$$(T_{A_1, A_2}f)(x) = \int_{\mathbb{R}} f(y) K_{A_1, A_2}(x, y) dy, \quad \forall f \in L^p(\mathbb{R}).$$

Proof. Since $g, \hat{g} \in L^1(\mathbb{R})$, we have $g \in L^p(\mathbb{R})$ for all $1 < p < \infty$. Hence $F_g f$ is well defined for any $f \in L^p(\mathbb{R})$. We have

$$\begin{aligned} (T_{A_1, A_2}f)(x) &= \int_{-A_1}^{A_2} (F_g f)(x - x_0, \omega) e^{ix\omega} d\omega \\ &= \int_{-A_1}^{A_2} d\omega \int_{\mathbb{R}} f(y) \overline{g(y - x + x_0)} e^{-iy\omega} e^{ix\omega} dy \\ &= \int_{\mathbb{R}} dy \int_{-A_1}^{A_2} f(y) \overline{g(y - x + x_0)} e^{-i(y-x)\omega} d\omega \\ &= \int_{\mathbb{R}} f(y) K_{A_1, A_2}(x, y) dy, \end{aligned}$$

where Fubini's theorem is used. This completes the proof. \square

The pointwise convergence of Fourier series is a deep result in harmonic analysis. Carleson proved that the Fourier series of a function in $L^2[-\pi, \pi]$ is convergent almost everywhere [2]. Hunt [9] extended this result to $L^p[-\pi, \pi]$ for $1 < p < \infty$. And Kenig and Tomas [10] proved the pointwise convergence of Fourier integral on $L^p(\mathbb{R})$. For our purpose, we cite the Carleson-Hunt theorem in the following form.

Proposition 2.5 *For $A > 0$ and $1 < p < \infty$, define*

$$(S_A f)(x) = \int_{\mathbb{R}} f(y) \frac{\sin A(x - y)}{\pi(x - y)} dy, \quad f \in L^p(\mathbb{R}). \quad (2.4)$$

Then S_A is a bounded linear operator on $L^p(\mathbb{R})$ and there exists some constant C_p such that

$$\left\| \sup_{A > 0} |(S_A f)(x)| \right\|_p \leq C_p \|f\|_p.$$

The Fourier multiplier is a useful tool in the study of Fourier transform. The following result on the Fourier multiplier is useful in studying the convergence of T_{A_1, A_2} .

Proposition 2.6 ([6, Corollary 3.8]) *Suppose that h is a function of bounded variation on \mathbb{R} and that $(Tf) = h \cdot \hat{f}$ for $f \in L^2(\mathbb{R})$. Then T can be extended to an operator on $L^p(\mathbb{R})$, $1 < p < \infty$ and*

$$\|Tf\|_p \leq C_p V_h \|f\|_p, \quad \forall f \in L^p(\mathbb{R}),$$

where V_h is the total variation of h on \mathbb{R} and C_p is a constant depending only on p .

The following lemma shows that $T_{A_1, A_2}f$ converges to f in $L^p(\mathbb{R})$ whenever f is in $C_c^1(\mathbb{R})$, the space of all continuous differentiable functions which are compactly supported.

Lemma 2.7 *For any $f \in C_c^1(\mathbb{R})$, we have*

$$\lim_{A_1, A_2 \rightarrow \infty} \|T_{A_1, A_2}f - 2\pi \overline{g(x_0)}f\|_p = 0, \quad 1 < p \leq \infty. \quad (2.5)$$

Proof. Fix some $f \in C_c^1(\mathbb{R})$. Suppose that $\text{supp } f \subset [-\Omega, \Omega]$ for some constant $\Omega > 0$. Since $f, f' \in L^2(\mathbb{R})$, we have $\hat{f} \in L^1(\mathbb{R})$. By Lemma 2.3,

$$\lim_{A_1, A_2 \rightarrow \infty} \|T_{A_1, A_2}f - 2\pi \overline{g(x_0)}f\|_\infty = 0. \quad (2.6)$$

Next we assume that $1 < p < \infty$. By (2.6), we have

$$\lim_{A_1, A_2 \rightarrow \infty} \|(T_{A_1, A_2}f - 2\pi \overline{g(x_0)}f) \cdot \chi_{[-2\Omega, 2\Omega]}\|_p = 0. \quad (2.7)$$

On the other hand, put

$$K(x, y) = \frac{4\|g\|_\infty}{|x - y|}, \quad x \neq y.$$

By Minkovski's inequality, we have

$$\begin{aligned} & \left(\int_{|x| \geq 2\Omega} \left| \int_{|y| \leq \Omega} K(x, y) |f(y)| dy \right|^p dx \right)^{1/p} \\ & \leq \int_{|y| \leq \Omega} |f(y)| \left(\int_{|x| \geq 2\Omega} |K(x, y)|^p dx \right)^{1/p} dy \\ & \leq 4\|g\|_\infty \int_{|y| \leq \Omega} |f(y)| \left(\int_{|x| > 2\Omega} \frac{dx}{|x - y|^p} \right)^{1/p} dy \\ & = M_p \|f\|_1 \\ & \leq M_p (2\Omega)^{1/p'} \|f\|_p, \end{aligned}$$

where M_p is a constant and $1/p + 1/p' = 1$. Note that

$$|(T_{A_1, A_2}f)(x)| \leq \int_{|y| \leq \Omega} |K_{A_1, A_2}(x, y) f(y)| dy \leq \int_{|y| \leq \Omega} K(x, y) |f(y)| dy.$$

By the dominated convergence theorem, we have

$$\lim_{A \rightarrow \infty} \|(T_{A_1, A_2} f - 2\pi \overline{g(x_0)} f) \cdot \chi_{\mathbb{R} \setminus [-2\Omega, 2\Omega]}\|_p = 0. \quad (2.8)$$

Now the conclusion follows by combining (2.7) and (2.8). \square

We are now ready to give the proof of the main result.

Proof of Theorem 1.1. First, we prove the convergence in $L^p(\mathbb{R})$.

For any $f \in L^2(\mathbb{R})$, by (2.2), we have

$$(T_{A_1, A_2} f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y) e^{iyx} dy \int_{y-A_2}^{y+A_1} \overline{\hat{g}(\omega)} e^{-ix_0 \omega} d\omega. \quad (2.9)$$

Hence

$$(T_{A_1, A_2} f)^\wedge(y) = h_{A_1, A_2}(y) \hat{f}(y),$$

where

$$h_{A_1, A_2}(y) = \int_{y-A_2}^{y+A_1} \overline{\hat{g}(\omega)} e^{-ix_0 \omega} d\omega.$$

Obviously, h_{A_1, A_2} is of bounded variation on \mathbb{R} and $V_{h_{A_1, A_2}} \leq 2\|\hat{g}\|_1$.

By Lemma 2.4 and Proposition 2.6, T_{A_1, A_2} is a bounded linear operator on $L^p(\mathbb{R})$ and

$$\|T_{A_1, A_2} f\| \leq 2C_p \|\hat{g}\|_1 \|f\|_p, \quad \forall f \in L^p(\mathbb{R}). \quad (2.10)$$

Fix some $f \in L^p(\mathbb{R})$. For any $\varepsilon > 0$, there is some $\tilde{f} \in C_c^1(\mathbb{R})$ such that $\|f - \tilde{f}\|_p < \varepsilon$. By Lemma 2.7, we can find some $A_0 > 0$ such that for any $A_1, A_2 > A_0$,

$$\|T_{A_1, A_2} \tilde{f} - 2\pi \overline{g(x_0)} \tilde{f}\|_p < \varepsilon.$$

Consequently,

$$\begin{aligned} \|T_{A_1, A_2} f - 2\pi \overline{g(x_0)} f\|_p &\leq \|T_{A_1, A_2} (f - \tilde{f})\|_p + \|T_{A_1, A_2} \tilde{f} - 2\pi \overline{g(x_0)} \tilde{f}\|_p \\ &\quad + 2\pi |g(x_0)| \cdot \|f - \tilde{f}\|_p \\ &\leq (2C_p \|\hat{g}\|_1 + 2\pi |g(x_0)| + 1) \varepsilon. \end{aligned}$$

Hence

$$\lim_{A_1, A_2 \rightarrow \infty} \|T_{A_1, A_2} f - 2\pi \overline{g(x_0)} f\|_p = 0, \quad \forall f \in L^p(\mathbb{R}).$$

Next we consider the pointwise convergence. For $A > 0$, let S_A be defined by (2.4). Then S_A is a bounded linear operator on $L^p(\mathbb{R})$ and

$$(S_A f)^\wedge = \hat{f} \cdot \chi_{[-A, A]}, \quad f \in L^2(\mathbb{R}).$$

For $f \in L^2 \cap L^p(\mathbb{R})$, define

$$(\tilde{T}_A f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\hat{g}(\omega)} e^{-ix_0 \omega} (M_{-\omega} S_A M_{\omega} f)(x) d\omega,$$

where the operator M_{ω} is defined by

$$(M_{\omega} f)(x) = e^{-ix\omega} f(x).$$

Since $\hat{g} \in L^1(\mathbb{R})$ and $L^2 \cap L^p(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, \tilde{T}_A can be extended to a bounded linear operator on $L^p(\mathbb{R})$.

On the other hand, for any $f \in L^2 \cap L^p(\mathbb{R})$, we see from (2.2) that

$$\begin{aligned}
(T_A f)(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\hat{g}(\omega)} e^{-ix_0\omega} d\omega \int_{\omega-A}^{\omega+A} \hat{f}(y) e^{iyx} dy \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\hat{g}(\omega)} e^{-ix_0\omega} d\omega \int_{-A}^A \hat{f}(y+\omega) e^{iyx} e^{ix\omega} dy \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\hat{g}(\omega)} e^{-ix_0\omega} (M_{-\omega} S_A M_{\omega} f)(x) d\omega \\
&= (\tilde{T}_A f)(x).
\end{aligned}$$

Using the density of $L^2 \cap L^p(\mathbb{R})$ again, we get that $T_A = \tilde{T}_A$ on $L^p(\mathbb{R})$. Hence

$$(T_A f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\hat{g}(\omega)} e^{-ix_0\omega} (M_{-\omega} S_A M_{\omega} f)(x) d\omega, \quad \forall f \in L^p(\mathbb{R}). \quad (2.11)$$

It follows that

$$\sup_{A>0} |(T_A f)(x)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{g}(\omega)| \cdot \sup_{A>0} |(M_{-\omega} S_A M_{\omega} f)(x)| d\omega.$$

By Minkovski's inequality and Proposition 2.5, we have

$$\begin{aligned}
\left\| \sup_{A>0} |(T_A f)(x)| \right\|_p &\leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{g}(\omega)| \cdot \left\| \sup_{A>0} |(M_{-\omega} S_A M_{\omega} f)(x)| \right\|_p d\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{g}(\omega)| \cdot \left\| \sup_{A>0} |(S_A M_{\omega} f)(x)| \right\|_p d\omega \\
&\leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{g}(\omega)| \cdot C_p \|M_{\omega} f\|_p d\omega \\
&= \frac{C_p}{2\pi} \|\hat{g}\|_1 \|f\|_p, \quad \forall f \in L^p(\mathbb{R}).
\end{aligned} \quad (2.12)$$

Fix some $f \in L^p(\mathbb{R})$. For any $\varepsilon > 0$, we can find some $\tilde{f} \in C_c^1(\mathbb{R})$ such that

$$\|f - \tilde{f}\|_p < \varepsilon.$$

By Lemma 2.7, we have

$$\lim_{A \rightarrow \infty} \|T_A \tilde{f} - 2\pi \overline{g(x_0)} \tilde{f}\|_{\infty} = 0.$$

Note that $T_A f$ is continuous on \mathbb{R} , thanks to Lemma 2.4. We have

$$\lim_{A \rightarrow \infty} \sup_{x \in \mathbb{R}} |(T_A \tilde{f})(x) - 2\pi \overline{g(x_0)} \tilde{f}(x)| = 0.$$

Hence

$$\limsup_{A, A' \rightarrow \infty} |(T_A \tilde{f})(x) - (T_{A'} \tilde{f})(x)| = 0, \quad \forall x \in \mathbb{R}.$$

It follows that

$$\begin{aligned}
& \left\| \limsup_{A, A' \rightarrow \infty} |(T_A f)(x) - (T_{A'} f)(x)| \right\|_p \\
& \leq \left\| \limsup_{A \rightarrow \infty} |(T_A(f - \tilde{f}))(x)| \right\|_p + \left\| \limsup_{A, A' \rightarrow \infty} |(T_A \tilde{f})(x) - (T_{A'} \tilde{f})(x)| \right\|_p \\
& \quad + \left\| \limsup_{A' \rightarrow \infty} |(T_{A'}(f - \tilde{f}))(x)| \right\|_p \\
& \leq 2 \left\| \sup_{A > 0} |(T_A(f - \tilde{f}))(x)| \right\|_p \\
& \leq \frac{C_p}{\pi} \|\hat{g}\|_1 \|f - \tilde{f}\|_p \quad (\text{using (2.12)}) \\
& < \frac{C_p}{\pi} \|\hat{g}\|_1 \cdot \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we have

$$\left\| \limsup_{A, A' \rightarrow \infty} |(T_A f)(x) - (T_{A'} f)(x)| \right\|_p = 0.$$

Hence the limit $\lim_{A \rightarrow \infty} (T_A f)(x)$ exists almost everywhere. Since $T_A f$ tends to $2\pi \overline{g(x_0)} f$ in $L^p(\mathbb{R})$, we have

$$\lim_{A \rightarrow \infty} (T_A f)(x) = 2\pi \overline{g(x_0)} f(x), \quad a.e.$$

This completes the proof. \square

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